Hexagonal circle packings and Doyle spirals

By Jos Leys, February 2005.
Some clarification for this webpage.

Peter Doyle discovered the fact that a set of 6 circles arranged around a central circle (a 'flower') with dimensions as indicated in Figure 1 below, may be extended to an infinite hexagonal circle packing of the plane. (which is possibly overlapping ,i.e. not coherent).
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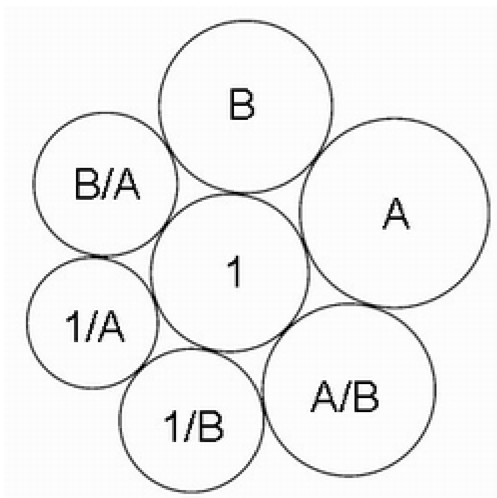

Figure 1

With this knowledge, it is possible to construct an infinite amount of circle arrangements, as proven by Stephenson et.al [1].

Let *C* be a circle with radius r, and center at 1 on the real axis. Let *A* and *B* be complex numbers.

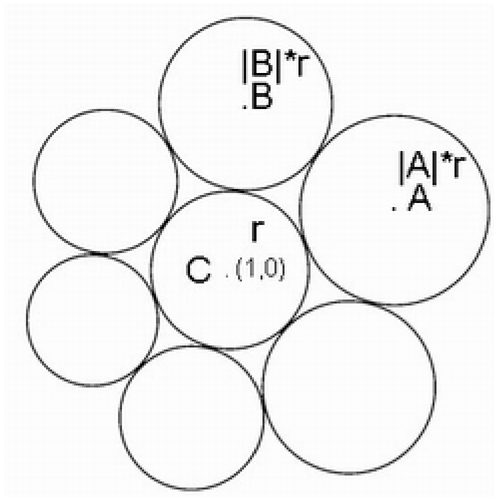

Figure 2

There are circles with radius $|A|*r$, centered at A, and circles with radius $|B|*r$ centered at B so that these circles are tangent to C and to each other. (Figure 2)

Putting $A=f.exp(i*\theta)$ and $B= g.exp(i*\varphi)$, the following equations apply :

$$r\^{}2=\frac{(g\^{}2-2*g*cos(\varphi)+1)}{(1+g)\^{}2}$$

$$r\^{}2=\frac{(f\^{}2-2*f*cos(\theta)+1)}{(1+f)\^{}2}$$

$$r\^{}2=\frac{(f\^{}2-2*f*g*cos(\theta-\varphi)+g\^{}2)}{(g+f)\^{}2}$$

Not all solutions to these equations will result in non-overlapping circle arrangements. We need an extra condition:

Let $B^q=A^p$ with q and p integers, then there is a solution for these equations for every p and q.

If $q=n*p$, then analytic solutions[2] exist for $n=1$, and $n=2$. Numerical methods are needed for other n.

A and B can be seen as transformations on the base circle C. The condition $B^q=A^p$ assures that the transformed circles coincide after q 'B' transformations or p 'A' transformations (Figure 3)

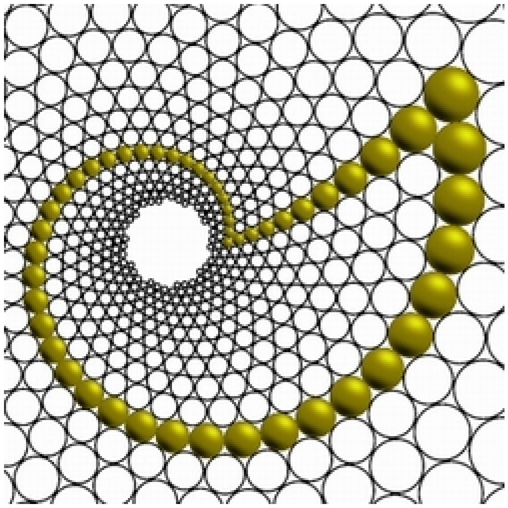

Figure 3

The circle packing patterns that are generated this way can be transformed onto the surface of a sphere by projection on the Riemann sphere, or through the use of a sphere inversion. Replacing all the circles by spheres of the same diameter produces images that are graphically more appealing. Both of these transform circles to circles and spheres to spheres. (Figure 4)

Figure 4

The above formulas can be adapted for the case whereby the circles overlap, with a given intersection angle, see Figure 5.

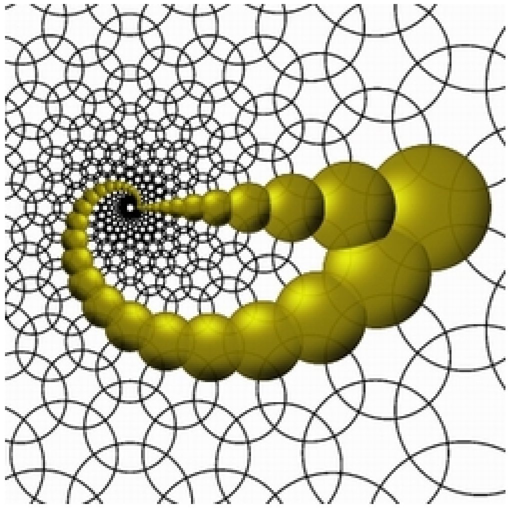

Figure 5

[1] Alan F. Beardon, Tomasz Dubejko, and Kenneth Stephenson, *Spiral hexagonal circle packings in the plane*, Geom. Dedicata **49** (1994), 39–70.

[2] David J. Wright, *Searching for the cusp*, Spaces of Kleinian Groups , Lond. Math. Soc. Lec. Notes **xxx**, 1–37. (see http://klein.math.okstate.edu/IndrasPearls/cusp.pdf)